(* Asymptotic expected number of hairpins in saturated structures, theta=1,p=1,using results from earlier version of paper *)

(*We first compute the dominant singularity and asymptotic number of saturated structures, using Drmota-Lalley-Woods Theorem. *)

```
eqn = {S == D0 + N0, D0 == z + z^2,
   N0 == R D0 + D0 z^2 + N0 z^2 + S D0 z^2 + S N0 z^2, R == D0 z^2 + N0 z^2 + R D0 z^2 + R N0 z^2};
Last[First[First[Solve[eqn, {S, D0, N0, R}]]]];
Eliminate[eqn, {N0, D0, R}]
F = S^3 z^4 + S (1 - z^2) + S^2 z^2 (-2 + z^2) - z (1 + z) + S;
NSolve[{F == S, D[F, S] == 1}, {z, S}]
z0 = 0.42468731042028074`;
y0 = 1.6568963458689865`;
dFdzOfz0S0 = D[F, z] /. {z → z0, S → y0}
d2FdyOfz0S0 = D[F, {S, 2}] /. {z → z0, S → y0}
c = Sqrt[z0 dFdzOfz0S0 / (2 Pi d2FdyOfz0S0)]
AsymptoticValue = c * (1 / z0)^n n^(-3 / 2)
```

(*Now, we compute mean,variance using Drmota's Theorem*)

```
(*S = z + z^2 + R*(z+z^2) + (z+z^2)u z^2 + (S-z-z^2)z^2 + S(z+z^2) u z^2 + S
   (S-z -z^2) z^2;
R = u z^2(z+z^2) + z^2 (S-z-z^2) +R(z+z^2)u z^2 + R(S-z-z^2) z^2;
*)
Clear[R, F, S, eqn, N0, D0, F, z0, y0, c];
(*Equation (5) from paper, written in form R = .... *)
sol =
  Solve[R == u z^2 (z + z^2) + z^2 (S - z - z^2) + R (z + z^2) u z^2 + R (S - z - z^2) z^2, R];
(* Now it follows that R = Last[Last[Last[sol]]]*)
(* Equation (4) from paper, written in form S = .... *)
rightSideOfS = z + z^2 + R * (z + z^2) +
  (z + z^2) u z^2 + (S - z - z^2) z^2 + S (z + z^2) u z^2 + S (S - z - z^2) z^2
(* Now replace R in right side of equation by previous expression "sol"*)
rightSideOfS /. R → Last[Last[Last[sol]]]
(* The following is the functional relation which satisfies y =
 F(z,y,u) where y is S*)
(* We now follow Drmota's paper "Systems of Functional Equations",
where his x,y,z are our z,S,u*)
F = %
f = (F /. S → s)
s - f
(* express over a common denominator*)
```

```
Together[s - f]
dfs = D[f, s]
1 - dfs
(* express over a common denominator*)
Together[1 - dfs]
(* a is numerator of s-f, and b is numerator of 1-dfs,
where both have identical denominators *)
a = (-1 + s z^2 - z^3 + u z^3 - z^4 + u z^4)
   (-s + z + z^2 + s z^2 + 2 s^2 z^2 - z^3 - 2 s z^3 + u z^3 + 2 s u z^3 - z^4 - 2 s z^4 - s^2 z^4 - s^3 z^4 + u z^4 + 2 s u z^4 +
     2 s z^5 + 2 s^2 z^5 - 2 s u z^5 - 2 s^2 u z^5 - z^6 + s z^6 + 2 s^2 z^6 + 2 u z^6 - 2 s^2 u z^6 - u^2 z^6 - s u^2 z^6 -
     2 z^7 - 2 s z^7 + 4 u z^7 + 4 s u z^7 - 2 u^2 z^7 - 2 s u^2 z^7 - z^8 - s z^8 + 2 u z^8 + 2 s u z^8 - u^2 z^8 - s u^2 z^8);
b = (1 - z^2 - 4 s z^2 + 2 z^3 - 3 u z^3 + 2 z^4 + 2 s z^4 + 5 s^2 z^4 - 3 u z^4 - 2 z^5 - 8 s z^5 + 2 u z^5 + 8 s u z^5 +
    z^6 - 8 s z^6 - s^2 z^6 - 2 s^3 z^6 - 4 u z^6 + 8 s u z^6 + 3 u^2 z^6 + 6 z^7 + 2 s z^7 + 5 s^2 z^7 - 12 u z^7 - 2 s u z^7 -
    5 s^2 u z^7 + 6 u^2 z^7 + 2 z^8 - 2 s z^8 + 5 s^2 z^8 - 4 u z^8 + 6 s u z^8 - 5 s^2 u z^8 + 2 u^2 z^8 - 4 s u^2 z^8 - z^9 -
    8 s z^9 + u z^9 + 16 s u z^9 + u^2 z^9 - 8 s u^2 z^9 - u^3 z^9 + 2 z^10 - 4 s z^10 - 7 u z^10 + 8 s u z^10 + 8 u^2 z^10 -
    4 s u^2 z^10 - 3 u^3 z^10 + 3 z^11 - 9 u z^11 + 9 u^2 z^11 - 3 u^3 z^11 + z^12 - 3 u z^12 + 3 u^2 z^12 - u^3 z^12);
CellPrint["Now compute the resultant of numerators a,b to get relation between S,u,z"]

(*We compute resultant of numerators,
since we have Resultant[S-F,1-D[F,S],S]=0 which is not much use.*)
res = Resultant[a, b, s]
(* Replace z by z[u], a function of u *)
res /. z → z[u];
(* Now compute z'[u] *)
dres = D[%, u]
dzu = Last[Last[Last[Solve[dres == 0, z'[u]]]]];
(*z[1] equals rho, the dominant singularity *)
rho = 0.42468731042028074`
dzuEvaluatedAt1 = (dzu /. u → 1) /. z[1] → rho
CellPrint[
 "According to Drmota's Theorem 1, the mean equals -z'[1]/z[1], computed next. "]
mu = ((-dzu / z[u] /. u → 1) /. z[1] → rho)

(*Variance computation *)
CellPrint["Now compute variance, which by Drmota is -z''[1]/z[1] + mu^2 + mu"]
(* d2zu is z''[1] *)
d2zu = (((D[dzu, u] /. u → 1) /. z[1] → rho) /. z'[1] → dzuEvaluatedAt1)
var = -d2zu / rho + mu * mu + mu
```

$$S^3 z^4 + S\left(1 - z^2\right) + S^2 z^2 \left(-2 + z^2\right) == z\,(1 + z)$$

```
{{z → 3.2141, S → -0.587227}, {z → -0.854537, S → 0.988667},
 {z → 0.424687, S → 1.6569}, {z → -2.29493, S → -0.513379},
 {z → -1., S → 0.}, {z → -0.244657 + 0.5601 i, S → -0.741229 + 0.680476 i},
 {z → -0.244657 - 0.5601 i, S → -0.741229 - 0.680476 i}}
```

-5.68551

-0.33299

1.07427

$$\frac{1.07427 \times 2.35467^n}{n^{3/2}}$$

$$z + z^2 + z^2 \left(S - z - z^2\right) + S z^2 \left(S - z - z^2\right) + R\left(z + z^2\right) + u z^2 \left(z + z^2\right) + S u z^2 \left(z + z^2\right)$$

$$z + z^2 + z^2 \left(S - z - z^2\right) + S z^2 \left(S - z - z^2\right) + u z^2 \left(z + z^2\right) + S u z^2 \left(z + z^2\right) + \frac{\left(z + z^2\right)\left(-S z^2 + z^3 - u z^3 + z^4 - u z^4\right)}{-1 + S z^2 - z^3 + u z^3 - z^4 + u z^4}$$

$$z + z^2 + z^2 \left(S - z - z^2\right) + S z^2 \left(S - z - z^2\right) + u z^2 \left(z + z^2\right) + S u z^2 \left(z + z^2\right) + \frac{\left(z + z^2\right) \left(-S z^2 + z^3 - u z^3 + z^4 - u z^4\right)}{-1 + S z^2 - z^3 + u z^3 - z^4 + u z^4}$$

$$z + z^2 + z^2 \left(s - z - z^2\right) + s z^2 \left(s - z - z^2\right) + u z^2 \left(z + z^2\right) + s u z^2 \left(z + z^2\right) + \frac{\left(z + z^2\right) \left(-s z^2 + z^3 - u z^3 + z^4 - u z^4\right)}{-1 + s z^2 - z^3 + u z^3 - z^4 + u z^4}$$

$$s - z - z^2 - z^2 \left(s - z - z^2\right) - s z^2 \left(s - z - z^2\right) - u z^2 \left(z + z^2\right) - s u z^2 \left(z + z^2\right) - \frac{\left(z + z^2\right) \left(-s z^2 + z^3 - u z^3 + z^4 - u z^4\right)}{-1 + s z^2 - z^3 + u z^3 - z^4 + u z^4}$$

$$\frac{1}{-1 + s z^2 - z^3 + u z^3 - z^4 + u z^4} \left(-s + z + z^2 + s z^2 + 2 s^2 z^2 - z^3 - 2 s z^3 + u z^3 + 2 s u z^3 - z^4 - 2 s z^4 - s^2 z^4 - s^3 z^4 + u z^4 + 2 s u z^4 + 2 s z^5 + 2 s^2 z^5 - 2 s u z^5 - 2 s^2 u z^5 - z^6 + s z^6 + 2 s^2 z^6 + 2 u z^6 - 2 s^2 u z^6 - u^2 z^6 - s u^2 z^6 - 2 z^7 - 2 s z^7 + 4 u z^7 + 4 s u z^7 - 2 u^2 z^7 - 2 s u^2 z^7 - z^8 - s z^8 + 2 u z^8 + 2 s u z^8 - u^2 z^8 - s u^2 z^8\right)$$

$$z^2 + s z^2 + z^2 \left(s - z - z^2\right) + u z^2 \left(z + z^2\right) - \frac{z^2 \left(z + z^2\right) \left(-s z^2 + z^3 - u z^3 + z^4 - u z^4\right)}{\left(-1 + s z^2 - z^3 + u z^3 - z^4 + u z^4\right)^2} - \frac{z^2 \left(z + z^2\right)}{-1 + s z^2 - z^3 + u z^3 - z^4 + u z^4}$$

$$1 - z^2 - s z^2 - z^2 \left(s - z - z^2\right) - u z^2 \left(z + z^2\right) + \frac{z^2 \left(z + z^2\right) \left(-s z^2 + z^3 - u z^3 + z^4 - u z^4\right)}{\left(-1 + s z^2 - z^3 + u z^3 - z^4 + u z^4\right)^2} + \frac{z^2 \left(z + z^2\right)}{-1 + s z^2 - z^3 + u z^3 - z^4 + u z^4}$$

$$\frac{1}{\left(-1 + s z^2 - z^3 + u z^3 - z^4 + u z^4\right)^2} \left(1 - z^2 - 4 s z^2 + 2 z^3 - 3 u z^3 + 2 z^4 + 2 s z^4 + 5 s^2 z^4 - 3 u z^4 - 2 z^5 - 8 s z^5 + 2 u z^5 + 8 s u z^5 + z^6 - 8 s z^6 - s^2 z^6 - 2 s^3 z^6 - 4 u z^6 + 8 s u z^6 + 3 u^2 z^6 + 6 z^7 + 2 s z^7 + 5 s^2 z^7 - 12 u z^7 - 2 s u z^7 - 5 s^2 u z^7 + 6 u^2 z^7 + 2 z^8 - 2 s z^8 + 5 s^2 z^8 - 4 u z^8 + 6 s u z^8 - 5 s^2 u z^8 + 2 u^2 z^8 - 4 s u^2 z^8 - z^9 - 8 s z^9 + u z^9 + 16 s u z^9 + u^2 z^9 - 8 s u^2 z^9 - u^3 z^9 + 2 z^{10} - 4 s z^{10} - 7 u z^{10} + 8 s u z^{10} + 8 u^2 z^{10} - 4 s u^2 z^{10} - 3 u^3 z^{10} + 3 z^{11} - 9 u z^{11} + 9 u^2 z^{11} - 3 u^3 z^{11} + z^{12} - 3 u z^{12} + 3 u^2 z^{12} - u^3 z^{12}\right)$$

Now compute the resultant of numerators a,b to get relation between S,u,z

$$\left(z^9 + z^{10}\right) \left(4 z^{18} + 9 z^{19} - z^{20} - 29 z^{21} - 12 u z^{21} - 57 z^{22} - 38 u z^{22} - 46 z^{23} - 46 u z^{23} + 18 z^{24} - 58 u z^{24} + 12 u^2 z^{24} + 91 z^{25} - 132 u z^{25} + 49 u^2 z^{25} + 135 z^{26} - 218 u z^{26} + 87 u^2 z^{26} + 139 z^{27} - 246 u z^{27} + 111 u^2 z^{27} - 4 u^3 z^{27} + 117 z^{28} - 242 u z^{28} + 145 u^2 z^{28} - 20 u^3 z^{28} + 88 z^{29} - 216 u z^{29} + 168 u^2 z^{29} - 40 u^3 z^{29} + 52 z^{30} - 144 u z^{30} + 132 u^2 z^{30} - 40 u^3 z^{30} + 20 z^{31} - 60 u z^{31} + 60 u^2 z^{31} - 20 u^3 z^{31} + 4 z^{32} - 12 u z^{32} + 12 u^2 z^{32} - 4 u^3 z^{32}\right)$$

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(4 z[u]^18 + 9 z[u]^19 - z[u]^20 - 29 z[u]^21 - 12 u z[u]^21 - 57 z[u]^22 - 38 u z[u]^22 - 46 z[u]^23 - 46 u z[u]^23 +
    18 z[u]^24 - 58 u z[u]^24 + 12 u^2 z[u]^24 + 91 z[u]^25 - 132 u z[u]^25 + 49 u^2 z[u]^25 + 135 z[u]^26 -
    218 u z[u]^26 + 87 u^2 z[u]^26 + 139 z[u]^27 - 246 u z[u]^27 + 111 u^2 z[u]^27 - 4 u^3 z[u]^27 +
    117 z[u]^28 - 242 u z[u]^28 + 145 u^2 z[u]^28 - 20 u^3 z[u]^28 + 88 z[u]^29 - 216 u z[u]^29 +
    168 u^2 z[u]^29 - 40 u^3 z[u]^29 + 52 z[u]^30 - 144 u z[u]^30 + 132 u^2 z[u]^30 - 40 u^3 z[u]^30 + 20 z[u]^31 -
    60 u z[u]^31 + 60 u^2 z[u]^31 - 20 u^3 z[u]^31 + 4 z[u]^32 - 12 u z[u]^32 + 12 u^2 z[u]^32 - 4 u^3 z[u]^32)
  (9 z[u]^8 Derivative[1][z][u] + 10 z[u]^9 Derivative[1][z][u]) +
 (z[u]^9 + z[u]^10) (-12 z[u]^21 - 38 z[u]^22 - 46 z[u]^23 - 58 z[u]^24 + 24 u z[u]^24 - 132 z[u]^25 +
    98 u z[u]^25 - 218 z[u]^26 + 174 u z[u]^26 - 246 z[u]^27 + 222 u z[u]^27 - 12 u^2 z[u]^27 - 242 z[u]^28 +
    290 u z[u]^28 - 60 u^2 z[u]^28 - 216 z[u]^29 + 336 u z[u]^29 - 120 u^2 z[u]^29 - 144 z[u]^30 +
    264 u z[u]^30 - 120 u^2 z[u]^30 - 60 z[u]^31 + 120 u z[u]^31 - 60 u^2 z[u]^31 - 12 z[u]^32 + 24 u z[u]^32 -
    12 u^2 z[u]^32 + 72 z[u]^17 Derivative[1][z][u] + 171 z[u]^18 Derivative[1][z][u] -
    20 z[u]^19 Derivative[1][z][u] - 609 z[u]^20 Derivative[1][z][u] -
    252 u z[u]^20 Derivative[1][z][u] - 1254 z[u]^21 Derivative[1][z][u] -
    836 u z[u]^21 Derivative[1][z][u] - 1058 z[u]^22 Derivative[1][z][u] -
    1058 u z[u]^22 Derivative[1][z][u] + 432 z[u]^23 Derivative[1][z][u] -
    1392 u z[u]^23 Derivative[1][z][u] + 288 u^2 z[u]^23 Derivative[1][z][u] +
    2275 z[u]^24 Derivative[1][z][u] - 3300 u z[u]^24 Derivative[1][z][u] +
    1225 u^2 z[u]^24 Derivative[1][z][u] + 3510 z[u]^25 Derivative[1][z][u] -
    5668 u z[u]^25 Derivative[1][z][u] + 2262 u^2 z[u]^25 Derivative[1][z][u] +
    3753 z[u]^26 Derivative[1][z][u] - 6642 u z[u]^26 Derivative[1][z][u] +
    2997 u^2 z[u]^26 Derivative[1][z][u] - 108 u^3 z[u]^26 Derivative[1][z][u] +
    3276 z[u]^27 Derivative[1][z][u] - 6776 u z[u]^27 Derivative[1][z][u] +
    4060 u^2 z[u]^27 Derivative[1][z][u] - 560 u^3 z[u]^27 Derivative[1][z][u] +
    2552 z[u]^28 Derivative[1][z][u] - 6264 u z[u]^28 Derivative[1][z][u] +
    4872 u^2 z[u]^28 Derivative[1][z][u] - 1160 u^3 z[u]^28 Derivative[1][z][u] +
    1560 z[u]^29 Derivative[1][z][u] - 4320 u z[u]^29 Derivative[1][z][u] +
    3960 u^2 z[u]^29 Derivative[1][z][u] - 1200 u^3 z[u]^29 Derivative[1][z][u] +
    620 z[u]^30 Derivative[1][z][u] - 1860 u z[u]^30 Derivative[1][z][u] +
    1860 u^2 z[u]^30 Derivative[1][z][u] - 620 u^3 z[u]^30 Derivative[1][z][u] +
    128 z[u]^31 Derivative[1][z][u] - 384 u z[u]^31 Derivative[1][z][u] +
    384 u^2 z[u]^31 Derivative[1][z][u] - 128 u^3 z[u]^31 Derivative[1][z][u])
```

```
0.424687
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```
-0.0523188
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According to Drmota's Theorem 1, the mean equals -z'[1]/z[1], computed next.

```
0.123194
```

Now compute variance, which by Drmota is -z''[1]/z[1] + mu^2 + mu

```
0.0442454
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```
0.0341867
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