(∗ Zuker grammar ∗)

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Clear[S, S0, P, P0, Q, rho, growth, Gnum, Gdenom, B, M, M1, U, z];

(* DSV equations from Zuker's treatment of dangles*)
eqn = {S == z + S z + (B + S B) (1 + 2 z + z^2),
   B == U z^2 + B z^2 + 2 U B z^2 + z^2 U^2 B + M M1 z^2 (1 + 2 z + z^2),
   M == M1 (1 + U + M), M1 == M1 z + B (1 + 2 z + z^2), U == z + U z};
Solve[eqn, {S, B, M, M1, U}];

(* S0 is solution for S the generating function for S *)
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$$\begin{aligned}S0 = \frac{1}{-z^2+2z^4-z^6}\Big(&-2z^4+z^5+z^6-\Big(1-3z+2z^2+z^3+z^4-z^5-z^6-\\&\sqrt{\left(1-6z+13z^2-14z^3+8z^4-23z^6-16z^7+19z^8+36z^9+15z^{10}-14z^{11}-15z^{12}-4z^{13}\right)}\Big)\Big/\\&\left(2\left(1-2z^2+4z^3+15z^4+20z^5+15z^6+6z^7+z^8\right)\right)+\\&\Big(z\Big(1-3z+2z^2+z^3+z^4-z^5-z^6-\sqrt{\left(1-6z+13z^2-14z^3+8z^4-23z^6-16z^7+19z^8+36z^9+15z^{10}-14z^{11}-15z^{12}-4z^{13}\right)}\Big)\Big)\Big/\left(1-2z^2+4z^3+15z^4+20z^5+15z^6+6z^7+z^8\right)-\\&\Big(z^2\Big(1-3z+2z^2+z^3+z^4-z^5-z^6-\sqrt{\left(1-6z+13z^2-14z^3+8z^4-23z^6-16z^7+19z^8+36z^9+15z^{10}-14z^{11}-15z^{12}-4z^{13}\right)}\Big)\Big)\Big/\left(2\left(1-2z^2+4z^3+15z^4+20z^5+15z^6+6z^7+z^8\right)\right)-\\&\Big(2z^3\Big(1-3z+2z^2+z^3+z^4-z^5-z^6-\sqrt{\left(1-6z+13z^2-14z^3+8z^4-23z^6-16z^7+19z^8+36z^9+15z^{10}-14z^{11}-15z^{12}-4z^{13}\right)}\Big)\Big)\Big/\left(1-2z^2+4z^3+15z^4+20z^5+15z^6+6z^7+z^8\right)-\\&\Big(3z^4\Big(1-3z+2z^2+z^3+z^4-z^5-z^6-\sqrt{\left(1-6z+13z^2-14z^3+8z^4-23z^6-16z^7+19z^8+36z^9+15z^{10}-14z^{11}-15z^{12}-4z^{13}\right)}\Big)\Big)\Big/\left(1-2z^2+4z^3+15z^4+20z^5+15z^6+6z^7+z^8\right)-\\&\Big(2z^5\Big(1-3z+2z^2+z^3+z^4-z^5-z^6-\sqrt{\left(1-6z+13z^2-14z^3+8z^4-23z^6-16z^7+19z^8+36z^9+15z^{10}-14z^{11}-15z^{12}-4z^{13}\right)}\Big)\Big)\Big/\left(1-2z^2+4z^3+15z^4+20z^5+15z^6+6z^7+z^8\right)-\\&\Big(z^6\Big(1-3z+2z^2+z^3+z^4-z^5-z^6-\sqrt{\left(1-6z+13z^2-14z^3+8z^4-23z^6-16z^7+19z^8+36z^9+15z^{10}-14z^{11}-15z^{12}-4z^{13}\right)}\Big)\Big)\Big/\left(2\left(1-2z^2+4z^3+15z^4+20z^5+15z^6+6z^7+z^8\right)\right)\Big);\end{aligned}$$

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(* Simplify S0 into expression S1 *)
Simplify[S0];

(* Now S=S0=G+H, where G=Gnum/Gdenom and H=-sqrt(P)/Gdenom*)
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$$Gnum = \left(1-3z+2z^2+z^3+5z^4+5z^5-3z^6-6z^7-2z^8\right);$$

$$Gdenom = \left(2\,(-1+z)^2\,z^2\,(1+z)^4\right);$$

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G = Gnum / Gdenom;
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$$P = \left(-(-1+z)^2\left(-1+4z-4z^2+2z^3-2z^5+19z^6+56z^7+74z^8+56z^9+23z^{10}+4z^{11}\right)\right);$$

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(* P0 is the factor to right of (-1+z)^2 in P*)
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$$P0 = \left(-1+4z-4z^2+2z^3-2z^5+19z^6+56z^7+74z^8+56z^9+23z^{10}+4z^{11}\right);$$

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(* Cannot symbolically solve P1==0; however get numerical solution *)
NSolve[{P0 == 0}, z];


(* dominant singularity ie smallest modulus root of eqn1,
with modulus smaller than 1, arising from (-1+z)^2 *)
rho = 0.32675583717839995`;


(* growth base of exponential is 3.060389092464863`.
 For some reason 1/rho doesn't replace rho by value.*)
growth = N[1 / 0.32675583717839995]
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(* Now rho is dominant singularity.Want to factor (1-z/rho) from P. This is done by
  computing Q=Series[P/(1-z/rho),{z,0,20}] and taking the resulting polynomial. *)
TaylorExpansionOfQ = Series[P / (1 - z / 0.3267558371783999), {z, 0, 20}];

(* P=(1-z/rho)*Q; ie Q = P/(1-z/rho)  *)
Q = {1 - 2.9396109075351364` z + 4.00364684248873` z^2 - 1.7472828731661014` z^3 +
    2.6526345535118097` z^4 + 8.118093853862916` z^5 + 1.8445258819680816` z^6 -
    10.355033110055729` z^7 - 12.690430382125669` z^8 - 2.837654720144201` z^9 +
    6.315672446278768` z^10 + 5.328415066222078` z^11 + 1.307023348635994` z^12};

(* Now S=
 G+H and H=-sqrt(1-z/rho)*sqrt(Q)/Gdenom. We now evaluate G(rho) to obtain G(rho)=
   1.1252982128552533` *)
G /. (z → 0.32675583717839995`);

(* Evaluate Gdenom(rho) = 0.29990653296117986` *)
Gdenom /. {z → 0.32675583717839995`};

(* Now evaluate Q at rho,
 i.e. Q/.{z→0.32675583717839995`}. We get Q(rho) = 0.4629228108359397`.} *)
{1 - 2.9396109075351364` z + 4.00364684248873` z^2 -
    1.7472828731661014` z^3 + 2.6526345535118097` z^4 + 8.118093853862916` z^5 +
    1.8445258819680816` z^6 - 10.355033110055729` z^7 - 12.690430382125669` z^8 -
    2.837654720144201` z^9 + 6.315672446278768` z^10 + 5.328415066222078` z^11 +
    1.307023348635994` z^12} /. {z → 0.32675583717839995`};

(* Now evaluate Sqrt[Q(rho)] to get 0.6803843111330093` *)

Sqrt[0.4629228108359397`];

(* It follows that S(rho) sim G(rho)-sqrt(1-z/rho)*sqrt(Q(rho))/Gdenom(rho) *)
Sfinal = 1.1252982128552533` -
  (Sqrt[1 - z / 0.32675583717839995`] * 0.6803843111330093`) / 0.29990653296117986`
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$1.1253 - 2.26865 \sqrt{1 - 3.06039 z}$

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(* It follows that S(rho) sim G(rho)-K sqrt(1-z/rho),
where G(rho) = 1.1252982128552533` and K =2.268654518509868` *)

(* Gamma(-1/2) = -2 sqrt(pi) *)

Gamma[-1 / 2];

(* -K/Gamma(-1/2) = 0.639975624005909` *)

(-2.268654518509868`) / (-2 Sqrt[π]);

(* Asymptotic solution *)
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(* It follows that S_n sim 0.639975624005909` * n^{-3/2} * (3.060389092464863`)^n *)
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